An analytic solution to xkcd comic 135

Yingtong Li

2016-09-23

1 Introduction

Many students, scholars and enthusiasts of science, technology, engineering and mathematics may be familiar with the online comic *xkcd*. As a member of all of the above groups, I have followed xkcd, created by American cartoonist, physics graduate and former NASA roboticist Randall Munroe, and its quirky and often technical brand of geek humour for a great many years. [1]

Figure 1 shows an early xkcd comic from 2006, illustrating xkcd's unusual style of humour: As an example, velociraptor attacks, the subject of Figure 1, are a recurring inside joke on xkcd. What I find particularly intriguing about this particular comic, though, is that it poses a number of seemingly legitimate well-constructed questions. The first is a straightforward kinematics problem, while the third is impossible without the 'floor plan' referenced by the question, but the second question poses a significant challenge:

> You are at the centre of a 20 m equilateral triangle with a raptor at each corner. The top raptor has a wounded leg and is limited to a top speed of 10 m s^{-1}. The raptors will run toward you. At what angle should you run to maximize the time you stay alive?

The spirit of xkcd does not allow this question to be passed over without at least attempting a solution. (Indeed, in another comic, Munroe [3] jokingly identifies one of his hobbies as 'nerd sniping', distracting physicists and mathematicians with interesting problems.) Thus in this paper I will attempt to solve one interpretation of the above question.

While several solutions to this question have been made by simulating the motion of the human and the raptors using a computer program (courtesy of xkcd's computer-programming readership), I found these numerical solutions failed to satiate my mathematical curiosity. Thus in this paper I intend to apply my knowledge from HL IB Mathematics about solutions to differential equations to the problem and derive a solution analytically using these standard mathematical tools and concepts.

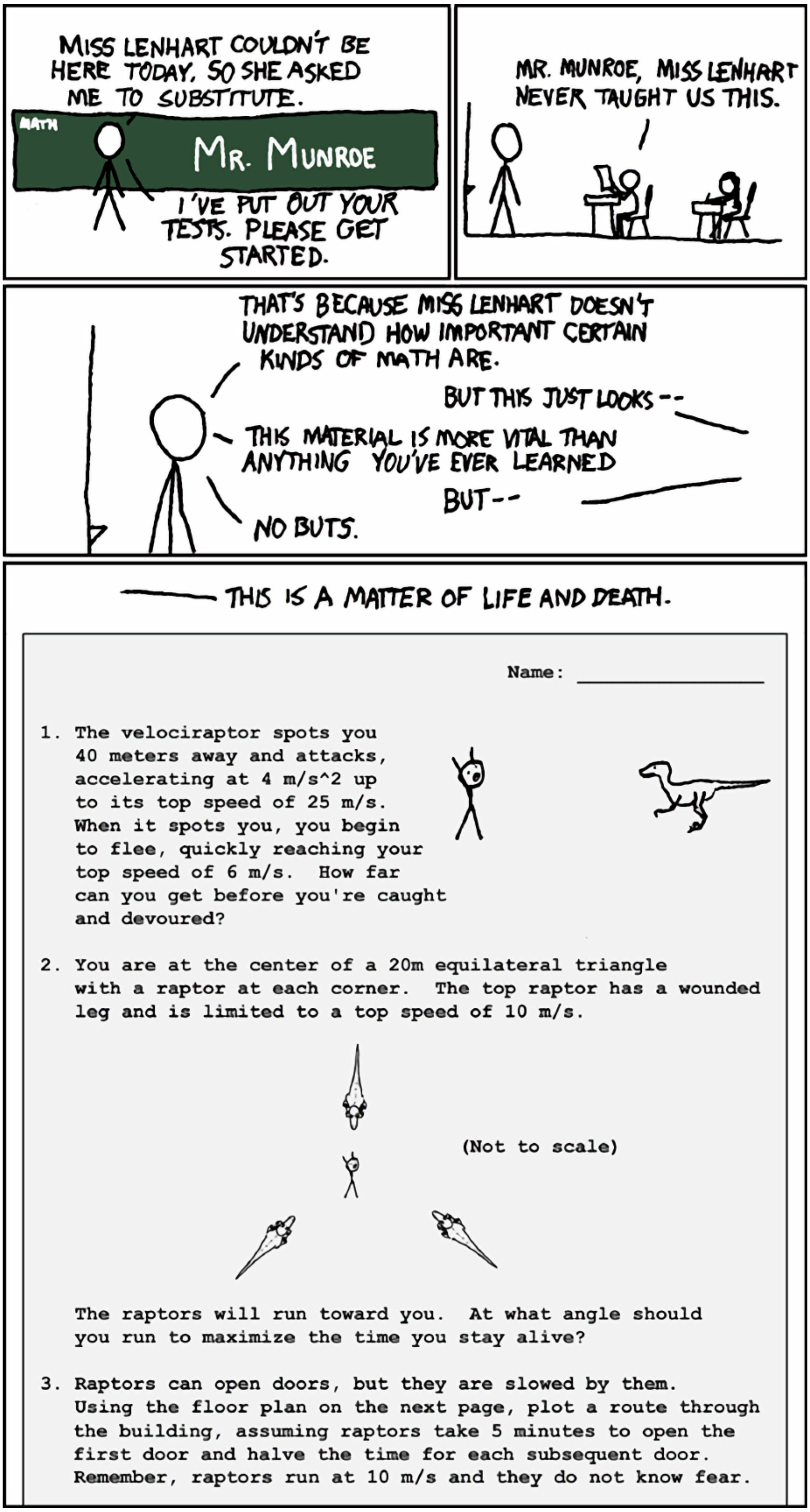

Figure 1: XKCD comic 135: 'Substitute'
Caption: 'YOU THINK THIS IS FUNNY?' [2]

2 Formulation of the question

Let us first consider the motion of the human. Based on the first sentence of question 2, we will suppose:

- A human is at the centre of an equilateral triangle of side length l m $= 20$ m with a raptor at each corner.

And since question 2 asks for a particular angle, we further suppose:

- The human begins at the origin, $(0, 0)$, and runs outward in a straight line away from the origin, making an angle φ with the y-axis, measured clockwise.

Finally, since question 1 mentions the human reaches its top speed of 6 m s^{-1} 'quickly', we will assume:

- The human runs at v_h m s^{-1} = 6 m s^{-1}, and has infinite acceleration up to this speed.

Unfortunately, the motion of the other component of the question, the velociraptors, is less clear. While the question states that 'The raptors will run towards' the human, each question suggests different values for the acceleration and maximum velocity of the raptors: Question 1 states that a raptor accelerates at 4 m s^{-2} up to a maximum of 25 m s^{-1}, whereas question 3 suggests a constant speed of 10 m s^{-1}, which is in turn contradicted by question 2, which implies that the top speed of an uninjured raptor is greater than 10 m s^{-1}, but does not quote specific values for acceleration or maximum speed.

In this paper, we will suppose that:

- One raptor, raptor 1, begins directly above the human (on the y-axis), runs at v_1 m s^{-1} = 10 m s^{-1}, and has infinite acceleration up to this speed.
- Two raptors begin 120° either side of the human, raptor 2 to the left and raptor 3 to the right, run at v_2 m s^{-1} = 25 m s^{-1}, and have infinite acceleration up to this speed.
- Each raptor runs towards the current position of the human at all times.

This is depicted in Figure 2.

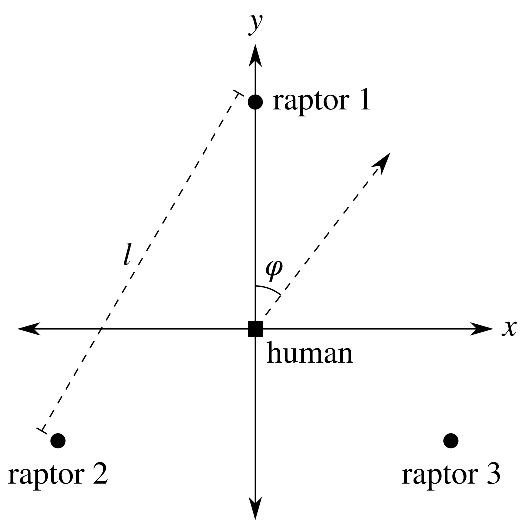

Figure 2: Diagram of the situation at the beginning of the question

Let k_1, k_2 and k_3 be the times taken for each raptor to individually catch the human, i.e. the time taken for the distance between the human and the raptor to reach 0, ignoring the other raptors. The question is:

- For what angle θ is the overall time of survival of the human, $k_{\text{min}} = \min(k_1, k_2, k_3)$, maximised?

3 Application of mathematics

Let us firstly consider the motion of a single raptor travelling at v_r m s^{-1}. The positions of the human and raptor over time, $\boldsymbol{s}_\text{h}$ and $\boldsymbol{s}_\text{r}$ respectively, can be approximated numerically using a geometric method similar to the Euler method as shown in Figure 3.

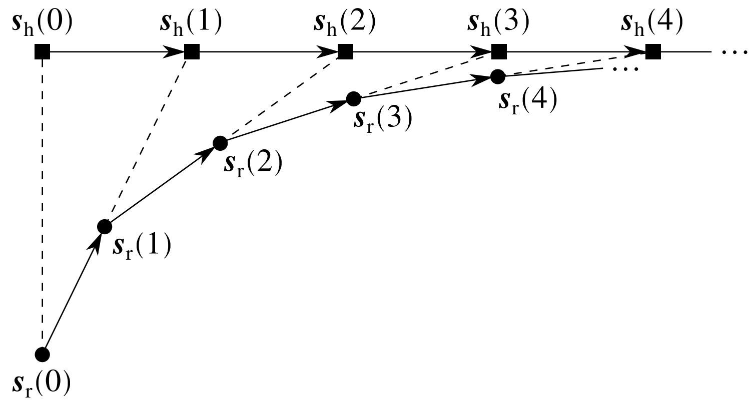

Figure 3: Geometric method for numerically approximating $\boldsymbol{s}_\text{h}$ and $\boldsymbol{s}_\text{r}$

In each time interval of Δt s (in Figure 3, 1 s), the situation can be modelled as the human travelling $v_\text{h}\Delta t$ m in its direction of motion, followed by the raptor moving $v_\text{r}\Delta t$ m towards the human's new position.

It is possible to solve for the equation of the raptor's motion (see Lloyd [4] and Mungan [5]), but as the question relates only to the distance between the raptor and the human, I have taken a different approach, focusing only on this distance D, and the angle θ as shown in Figure 4.

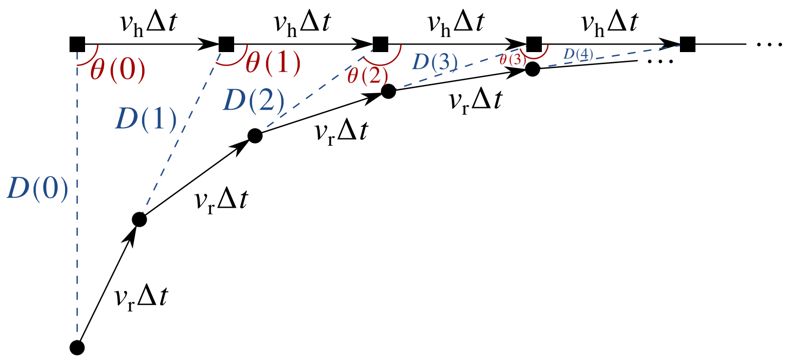

Figure 4: D and θ over time in Figure 3

Just as decreasing the step size in the Euler method increases the accuracy of the result, we will take the limit as $\Delta t \to 0$ to yield equations relating D and θ (in fact $\cos\theta$) over time. As the product of real-world physical kinematics, we may intuitively expect $D(t)$ and $\cos\theta(t)$ to be continuous and differentiable, so we will assume this is the case.

Consider the situation at a time t, as shown in Figure 5.

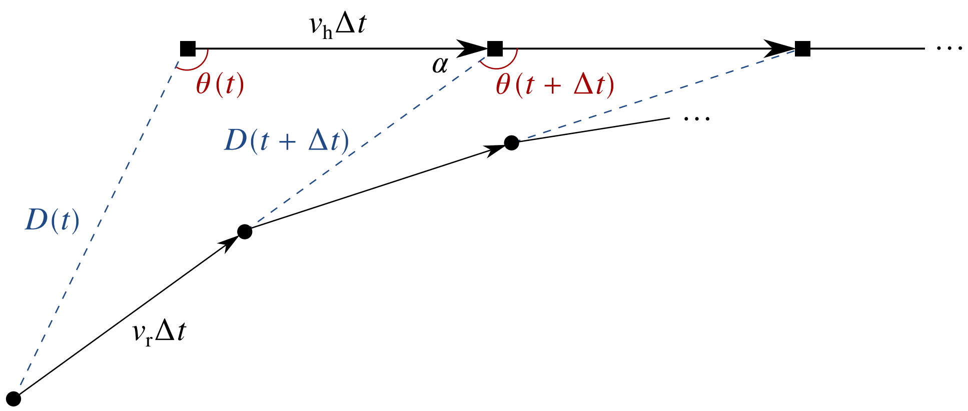

Figure 5: An extract from the model at time t

Applying the cosine rule to the triangle on the left with sides $D(t)$, $v_h\Delta t$ and $v_r\Delta t + D(t+\Delta t)$, using the angle $\theta(t)$,

$$(v_r\Delta t + D(t+\Delta t))^2 = (D(t))^2 + (v_h\Delta t)^2 - 2(D(t))(v_h\Delta t)\cos\theta(t)$$

Expanding and rearranging,

$$\begin{aligned} v_r{}^2(\Delta t)^2 + 2v_r\Delta t D(t+\Delta t) + (D(t+\Delta t))^2 &= (D(t))^2 + v_h{}^2(\Delta t)^2 - 2D(t)v_h\Delta t\cos\theta(t) \qquad (1)\\ (D(t+\Delta t))^2 - (D(t))^2 &= v_h{}^2(\Delta t)^2 - 2D(t)v_h\Delta t\cos\theta(t)\\ &\quad - v_r{}^2(\Delta t)^2 - 2v_r\Delta t D(t+\Delta t) \end{aligned}$$

Dividing both sides by Δt,

$$\frac{(D(t+\Delta t))^2 - (D(t))^2}{\Delta t} = v_h{}^2\Delta t - 2D(t)v_h\cos\theta(t) - v_r{}^2\Delta t - 2v_r D(t+\Delta t)$$

Taking the limit as $\Delta t \to 0$,

$$\lim_{\Delta t\to 0}\frac{(D(t+\Delta t))^2 - (D(t))^2}{\Delta t} = -2D(t)v_h\cos\theta(t) - 2v_r D(t)$$

Since $D(t)$ is assumed to be differentiable at t, from the first-principles definition of the derivative, the left-hand side is simply $\frac{\mathrm{d}}{\mathrm{d}t}D^2$. Substituting this in,

$$\frac{\mathrm{d}}{\mathrm{d}t}D^2 = -2Dv_h\cos\theta - 2v_r D$$

Applying the power rule to the left-hand side and writing in terms of $\frac{\mathrm{d}D}{\mathrm{d}t}$,

$$\begin{aligned} 2D\frac{\mathrm{d}D}{\mathrm{d}t} &= -2Dv_h\cos\theta - 2v_r D\\ \frac{\mathrm{d}D}{\mathrm{d}t} &= -v_h\cos\theta - v_r \qquad (2) \end{aligned}$$

Returning to Figure 5 and applying the cosine rule again, but using the angle $\alpha = 180° - \theta(t+\Delta t)$, so $\cos\alpha = -\cos\theta(t+\Delta t)$,

$$\begin{aligned} (D(t))^2 &= (v_r\Delta t + D(t+\Delta t))^2 + (v_h\Delta t)^2 + 2(v_r\Delta t + D(t+\Delta t))(v_h\Delta t)\cos\theta(t+\Delta t)\\ (D(t))^2 &= v_r^2(\Delta t)^2 + 2v_r\Delta t D(t+\Delta t) + (D(t+\Delta t))^2 + v_h{}^2(\Delta t)^2\\ &\quad + 2v_r(\Delta t)^2 v_h\cos\theta(t+\Delta t) + 2D(t+\Delta t)v_h\Delta t\cos\theta(t+\Delta t) \qquad (3) \end{aligned}$$

Adding Equation 1 and Equation 3 together and collecting like terms,

$$\begin{aligned}0 = {} & 2{v_\mathrm{h}}^2(\Delta t)^2 - 2D(t)v_\mathrm{h}\Delta t\cos\theta(t) + 2v_\mathrm{r}(\Delta t)^2 v_\mathrm{h}\cos\theta(t+\Delta t)\\ & + 2D(t+\Delta t)v_\mathrm{h}\Delta t\cos\theta(t+\Delta t)\end{aligned}$$

Dividing by $2v_\mathrm{h}\Delta t$ and rearranging,

$$D(t+\Delta t)\cos\theta(t+\Delta t) - D(t)\cos\theta(t) = -v_\mathrm{r}\Delta t\cos\theta(t+\Delta t) - v_\mathrm{h}\Delta t$$

Dividing by Δt and taking the limit as $\Delta t \to 0$,

$$\lim_{\Delta t\to 0}\frac{D(t+\Delta t)\cos\theta(t+\Delta t) - D(t)\cos\theta(t)}{\Delta t} = -v_\mathrm{r}\cos\theta(t) - v_\mathrm{h}$$

Once again, the left-hand side corresponds with the first-principles definition of a derivative, $\frac{\mathrm{d}}{\mathrm{d}t}D\cos\theta$. Substituting this in,

$$\frac{\mathrm{d}}{\mathrm{d}t}D\cos\theta = -v_\mathrm{r}\cos\theta - v_\mathrm{h}$$

Applying the product rule to the left-hand side,

$$\frac{\mathrm{d}D}{\mathrm{d}t}\cos\theta + D\frac{\mathrm{d}\cos\theta}{\mathrm{d}t} = -v_\mathrm{r}\cos\theta - v_\mathrm{h} \tag{4}$$

Together, Equations 2 and 4 form a first-order non-linear system of differential equations, the solution to which yields the equation for the distance between the human and raptor over time, $D(t)$.

Substituting 2 into 4,

$$\begin{aligned}(-v_\mathrm{h}\cos\theta - v_\mathrm{r})\cos\theta + D\frac{\mathrm{d}\cos\theta}{\mathrm{d}t} &= -v_\mathrm{r}\cos\theta - v_\mathrm{h}\\ -v_\mathrm{h}\cos^2\theta - v_\mathrm{r}\cos\theta + D\frac{\mathrm{d}\cos\theta}{\mathrm{d}t} &= -v_\mathrm{r}\cos\theta - v_\mathrm{h}\\ v_\mathrm{h}\cos^2\theta - v_\mathrm{h} &= D\frac{\mathrm{d}\cos\theta}{\mathrm{d}t}\end{aligned} \tag{5}$$

Note that by the chain rule, $\frac{\mathrm{d}\cos\theta}{\mathrm{d}t} = \frac{\mathrm{d}\cos\theta}{\mathrm{d}D}\frac{\mathrm{d}D}{\mathrm{d}t} = (-v_\mathrm{h}\cos\theta - v_\mathrm{r})\frac{\mathrm{d}\cos\theta}{\mathrm{d}D}$. Substituting this in,

$$v_\mathrm{h}\cos^2\theta - v_\mathrm{h} = D(-v_\mathrm{h}\cos\theta - v_\mathrm{r})\frac{\mathrm{d}\cos\theta}{\mathrm{d}D} \tag{6}$$

This is now a first-order separable differential equation in D and $\cos\theta$. Rearranging,

$$\frac{1}{D} = \frac{-v_\mathrm{h}\cos\theta - v_\mathrm{r}}{v_\mathrm{h}\cos^2\theta - v_\mathrm{h}}\frac{\mathrm{d}\cos\theta}{\mathrm{d}D}$$

Integrating with respect to D,

$$\begin{aligned}\int\frac{1}{D}\,\mathrm{d}D &= \int\frac{-v_\mathrm{h}\cos\theta - v_\mathrm{r}}{v_\mathrm{h}\cos^2\theta - v_\mathrm{h}}\,\mathrm{d}\cos\theta\\ \ln|D| + c_1 &= -\frac{1}{2}\int\frac{2v_\mathrm{h}\cos\theta}{v_\mathrm{h}\cos^2\theta - v_\mathrm{h}}\,\mathrm{d}\cos\theta - \int\frac{v_\mathrm{r}}{v_\mathrm{h}\cos^2\theta - v_\mathrm{h}}\,\mathrm{d}\cos\theta\\ \ln|D| + c_1 &= -\frac{1}{2}\ln\left|v_\mathrm{h}\cos^2\theta - v_\mathrm{h}\right| - \frac{v_\mathrm{r}}{v_\mathrm{h}}\int\frac{1}{\cos^2\theta - 1}\,\mathrm{d}\cos\theta\\ \ln|D| + c_1 &= -\frac{1}{2}\ln\left|v_\mathrm{h}\left(\cos^2\theta - 1\right)\right| - \frac{v_\mathrm{r}}{v_\mathrm{h}}\int\frac{1}{(\cos\theta - 1)(\cos\theta + 1)}\,\mathrm{d}\cos\theta\end{aligned}$$

To evaluate the final integral, note that $\frac{1}{(\cos\theta-1)(\cos\theta+1)} = \frac{1}{2(\cos\theta-1)} - \frac{1}{2(\cos\theta+1)}$. This result can be derived using partial fraction decomposition, and can be verified by expanding and simplifying. Substituting this result in,

$$\begin{aligned}
\ln|D| + c_1 &= -\frac{1}{2}\ln|v_\mathrm{h}(\cos\theta-1)(\cos\theta+1)| - \frac{v_\mathrm{r}}{v_\mathrm{h}}\int \frac{1}{2(\cos\theta-1)} - \frac{1}{2(\cos\theta+1)}\,\mathrm{d}\cos\theta \\
\ln|D| + c_1 &= -\frac{1}{2}\ln|v_\mathrm{h}||\cos\theta-1||\cos\theta+1| - \frac{v_\mathrm{r}}{2v_\mathrm{h}}\int \frac{1}{\cos\theta-1} - \frac{1}{\cos\theta+1}\,\mathrm{d}\cos\theta \\
\ln|D| + c_1 &= -\frac{1}{2}\ln|v_\mathrm{h}| - \frac{1}{2}\ln|\cos\theta-1| - \frac{1}{2}\ln|\cos\theta+1| - \frac{v_\mathrm{r}}{2v_\mathrm{h}}\ln|\cos\theta-1| + \frac{v_\mathrm{r}}{2v_\mathrm{h}}\ln|\cos\theta+1|
\end{aligned}$$

Since $v_\mathrm{h} > 0$, $D \geq 0$ and $-1 \leq \cos\theta \leq 1$, we can simplify the absolute value operations:

$$\begin{aligned}
\ln D + c_1 &= -\frac{1}{2}\ln v_\mathrm{h} - \frac{1}{2}\ln(1-\cos\theta) - \frac{1}{2}\ln(1+\cos\theta) - \frac{v_\mathrm{r}}{2v_\mathrm{h}}\ln(1-\cos\theta) + \frac{v_\mathrm{r}}{2v_\mathrm{h}}\ln(1+\cos\theta) \\
\ln D + c_2 &= \left(\frac{v_\mathrm{r}}{2v_\mathrm{h}} - \frac{1}{2}\right)\ln(1+\cos\theta) - \left(\frac{v_\mathrm{r}}{2v_\mathrm{h}} + \frac{1}{2}\right)\ln(1-\cos\theta) \quad (c_2 = c_1 + \ln v_\mathrm{h})
\end{aligned}$$

Raising both sides to base e and rearranging,

$$\begin{aligned}
\mathrm{e}^{\ln D + c_2} &= \mathrm{e}^{\left(\frac{v_\mathrm{r}}{2v_\mathrm{h}} - \frac{1}{2}\right)\ln(1+\cos\theta) - \left(\frac{v_\mathrm{r}}{2v_\mathrm{h}} + \frac{1}{2}\right)\ln(1-\cos\theta)} \\
D\mathrm{e}^{c_2} &= \frac{\mathrm{e}^{\left(\frac{v_\mathrm{r}}{2v_\mathrm{h}} - \frac{1}{2}\right)\ln(1+\cos\theta)}}{\mathrm{e}^{\left(\frac{v_\mathrm{r}}{2v_\mathrm{h}} + \frac{1}{2}\right)\ln(1-\cos\theta)}} \\
AD &= \frac{(1+\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}} - \frac{1}{2}}}{(1-\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}} + \frac{1}{2}}} \quad (A = \mathrm{e}^{c_2})
\end{aligned} \tag{7}$$

Now, as we wish to find the time $t = k$ when the human is caught by the raptor, i.e. when $D = 0$, the value of $\cos\theta$ at this time, $\cos\theta_k$, can be solved for by substituting $D = 0$ into Equation 7, and evaluating $\cos\theta_k$:

$$\begin{aligned}
0 &= \frac{(1+\cos\theta_k)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}} - \frac{1}{2}}}{(1-\cos\theta_k)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}} + \frac{1}{2}}} \\
0 &= 1 + \cos\theta_k \\
\cos\theta_k &= -1
\end{aligned}$$

Additionally, the constant of integration A can be solved for by substituting the the initial conditions D_0 and θ_0 into Equation 7, and evaluating A:

$$\begin{aligned}
AD_0 &= \frac{(1+\cos\theta_0)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}} - \frac{1}{2}}}{(1-\cos\theta_0)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}} + \frac{1}{2}}} \\
A &= \frac{(1+\cos\theta_0)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}} - \frac{1}{2}}}{D_0(1-\cos\theta_0)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}} + \frac{1}{2}}} \\
&= \frac{\left(\frac{1+\cos\theta_0}{1-\cos\theta_0}\right)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}} - \frac{1}{2}}}{D_0(1-\cos\theta_0)} \\
&= \frac{\left(\frac{2}{1-\cos\theta_0} - 1\right)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}} - \frac{1}{2}}}{D_0(1-\cos\theta_0)}
\end{aligned} \tag{8}$$

Now writing Equation 7 in terms of D,

$$D = \frac{(1+\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{1}{2}}}{A(1-\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}+\frac{1}{2}}}$$

Substituting this into Equation 5,

$$v_\mathrm{h}\cos^2\theta - v_\mathrm{h} = \frac{(1+\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{1}{2}}}{A(1-\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}+\frac{1}{2}}}\frac{\mathrm{d}\cos\theta}{\mathrm{d}t}$$

This is now a first-order separable differential equation in $\cos\theta$ and t. Rearranging,

$$\begin{aligned}
1 &= \frac{(1+\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{1}{2}}}{A(1-\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}+\frac{1}{2}}(v_\mathrm{h}\cos^2\theta - v_\mathrm{h})}\frac{\mathrm{d}\cos\theta}{\mathrm{d}t} \\
1 &= -\frac{(1+\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{1}{2}}}{Av_\mathrm{h}(1-\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}+\frac{1}{2}}(1-\cos\theta)(1+\cos\theta)}\frac{\mathrm{d}\cos\theta}{\mathrm{d}t} \\
1 &= -\frac{(1+\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{3}{2}}}{Av_\mathrm{h}(1-\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}+\frac{3}{2}}}\frac{\mathrm{d}\cos\theta}{\mathrm{d}t}
\end{aligned}$$

Integrating with respect to t from $t = 0$ to $t = k$, i.e. from $\cos\theta = \cos\theta_0$ to $\cos\theta = \cos\theta_k = -1$,

$$\begin{aligned}
\int_0^k 1\,\mathrm{d}t &= \int_{\cos\theta_0}^{-1} -\frac{(1+\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{3}{2}}}{Av_\mathrm{h}(1-\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}+\frac{3}{2}}}\,\mathrm{d}\cos\theta \\
k &= -\frac{1}{Av_\mathrm{h}}\int_{\cos\theta_0}^{-1}\frac{(1+\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{3}{2}}}{(1-\cos\theta)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}+\frac{3}{2}}}\,\mathrm{d}\cos\theta
\end{aligned}$$

Thus we have derived an expression for k as required. Evaluating the integral,

$$\begin{aligned}
k &= -\frac{1}{Av_\mathrm{h}}\int_{\cos\theta_0}^{-1}\frac{\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{3}{2}}}{(1-\cos\theta)^3}\,\mathrm{d}\cos\theta \\
&= -\frac{1}{Av_\mathrm{h}}\int_{\cos\theta_0}^{-1}\frac{\left(\frac{2}{1-\cos\theta}-1\right)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{3}{2}}}{1-\cos\theta}\frac{1}{(1-\cos\theta)^2}\,\mathrm{d}\cos\theta
\end{aligned}$$

Substituting $u = \frac{1}{1-\cos\theta}$ so $\frac{\mathrm{d}u}{\mathrm{d}\cos\theta} = -\frac{-1}{(1-\cos\theta)^2}$ and $\frac{1}{(1-\cos\theta)^2}\,\mathrm{d}\cos\theta = \mathrm{d}u$,

$$\begin{aligned}
k &= -\frac{1}{Av_\mathrm{h}}\int_{\frac{1}{1-\cos\theta_0}}^{\frac{1}{2}} u(2u-1)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{3}{2}}\,\mathrm{d}u \\
&= -\frac{1}{2Av_\mathrm{h}}\int_{\frac{1}{1-\cos\theta_0}}^{\frac{1}{2}} (2u-1+1)(2u-1)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{3}{2}}\,\mathrm{d}u \\
&= -\frac{1}{2Av_\mathrm{h}}\int_{\frac{1}{1-\cos\theta_0}}^{\frac{1}{2}} (2u-1)(2u-1)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{3}{2}} + (2u-1)^{\frac{v_\mathrm{r}}{2v_\mathrm{h}}-\frac{3}{2}}\,\mathrm{d}u
\end{aligned}$$

$$= -\frac{1}{4Av_h}\int_{\frac{1}{1-\cos\theta_0}}^{\frac{1}{2}} 2(2u-1)^{\frac{v_r}{2v_h}-\frac{1}{2}} + 2(2u-1)^{\frac{v_r}{2v_h}-\frac{3}{2}}\,\mathrm{d}u$$

$$= -\frac{1}{4Av_h}\left[\frac{(2u-1)^{\frac{v_r}{2v_h}+\frac{1}{2}}}{\frac{v_r}{2v_h}+\frac{1}{2}} + \frac{(2u-1)^{\frac{v_r}{2v_h}-\frac{1}{2}}}{\frac{v_r}{2v_h}-\frac{1}{2}}\right]_{\frac{1}{1-\cos\theta_0}}^{\frac{1}{2}}$$

$$= \frac{1}{4Av_h}\left(\frac{\left(\frac{2}{1-\cos\theta_0}-1\right)^{\frac{v_r}{2v_h}+\frac{1}{2}}}{\frac{v_r}{2v_h}+\frac{1}{2}} + \frac{\left(\frac{2}{1-\cos\theta_0}-1\right)^{\frac{v_r}{2v_h}-\frac{1}{2}}}{\frac{v_r}{2v_h}-\frac{1}{2}}\right) \quad \left(\text{provided } \frac{v_r}{2v_h}-\frac{1}{2}>0\right) \tag{9}$$

Substituting in Equation 8,

$$k = \frac{D_0(1-\cos\theta_0)}{4\left(\frac{2}{1-\cos\theta_0}-1\right)^{\frac{v_r}{2v_h}-\frac{1}{2}} v_h}\left(\frac{\left(\frac{2}{1-\cos\theta_0}-1\right)^{\frac{v_r}{2v_h}+\frac{1}{2}}}{\frac{v_r}{2v_h}+\frac{1}{2}} + \frac{\left(\frac{2}{1-\cos\theta_0}-1\right)^{\frac{v_r}{2v_h}-\frac{1}{2}}}{\frac{v_r}{2v_h}-\frac{1}{2}}\right)$$

$$= \frac{D_0(1-\cos\theta_0)}{4v_h}\left(\frac{\frac{2}{1-\cos\theta_0}-1}{\frac{v_r}{2v_h}+\frac{1}{2}} + \frac{1}{\frac{v_r}{2v_h}-\frac{1}{2}}\right)$$

$$= \frac{D_0}{4v_h}\left(\frac{1+\cos\theta_0}{\frac{v_r}{2v_h}+\frac{1}{2}} + \frac{1-\cos\theta_0}{\frac{v_r}{2v_h}-\frac{1}{2}}\right)$$

$$= D_0\left(\frac{1+\cos\theta_0}{2v_r+2v_h} + \frac{1-\cos\theta_0}{2v_r-2v_h}\right)$$

$$= D_0\left(\frac{2v_r-2v_h+2v_r\cos\theta_0-2v_h\cos\theta_0+2v_r+2v_h-2v_r\cos\theta_0-2v_h\cos\theta_0}{4{v_r}^2-4{v_h}^2}\right)$$

$$= D_0\left(\frac{v_r-v_h\cos\theta_0}{{v_r}^2-{v_h}^2}\right) \tag{10}$$

Returning to the situation described in the original question, the initial distances and angles between the raptors and the human are shown in Figure 6.

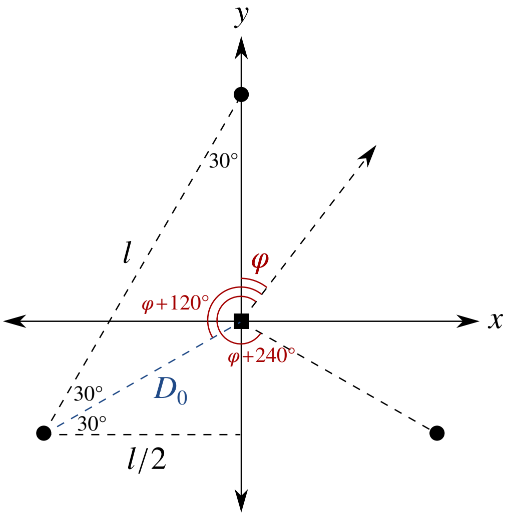

Figure 6: The initial distances and angles in the original question

The initial distance D_0 for each raptor is the same, $\frac{\frac{l}{2}}{\cos 30^\circ} = \frac{l}{\sqrt{3}} = \frac{20}{\sqrt{3}}$, while the initial angles θ_0 for raptors 1, 2 and 3 are φ, $\varphi + 120^\circ$ and $\varphi + 240^\circ$, respectively.

Thus, using Equation 10, given these initial conditions and the speeds of the human and raptors outlined in the original question, k_1, k_2 and k_3 are:

$$k_1 = \frac{20}{\sqrt{3}}\left(\frac{10 - 6\cos(\varphi)}{10^2 - 6^2}\right)$$
$$k_2 = \frac{20}{\sqrt{3}}\left(\frac{25 - 6\cos(\varphi + 120^\circ)}{25^2 - 6^2}\right)$$
$$k_3 = \frac{20}{\sqrt{3}}\left(\frac{25 - 6\cos(\varphi + 240^\circ)}{25^2 - 6^2}\right)$$

Figure 7 shows a graph of k_1, k_2 and k_3 against φ, and a graph of the overall time of survival $k_{\min}$, the lowest of the three functions for any given angle φ.

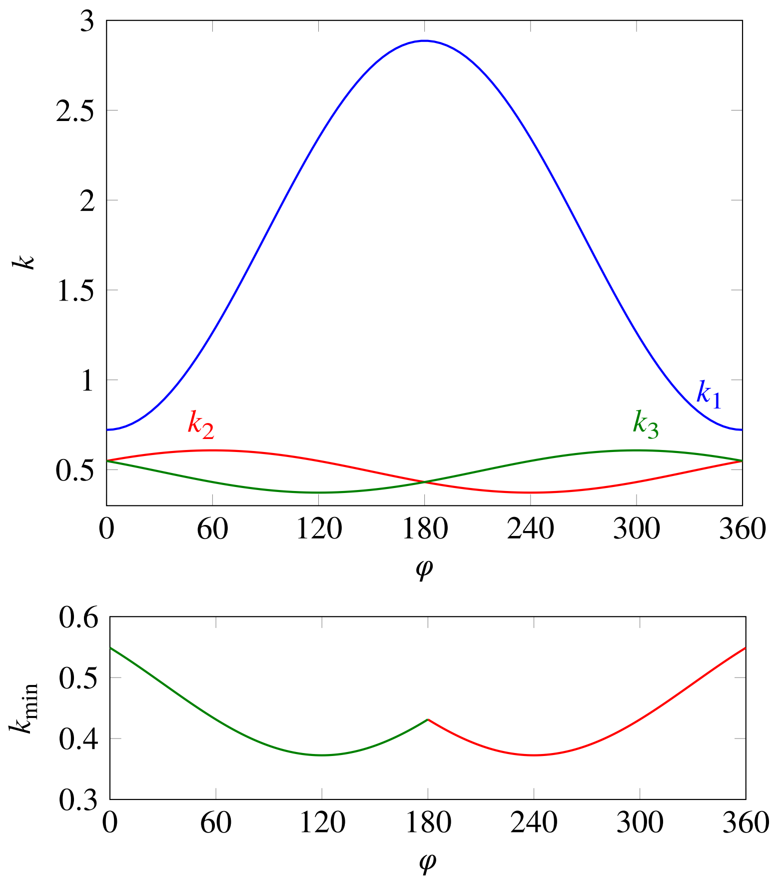

Figure 7: Graphs of time of survival k (seconds) against φ (degrees), given the conditions outlined in the original question

As Figure 7 shows, $k_{\min}$ is maximised when $\varphi = 0^\circ$, i.e. when the human runs directly towards raptor 1, the slower raptor. At this angle, $k_{\min} = k_2 = k_3 \approx 0.549$ (seconds).

This is a rather anticlimactic result, however the formula can be extended to other cases. If, for example, the speed of raptor 1 were to be 20 m s^{-1} instead of 10 m s^{-1}, the graph would be Figure 8, and the overall time of survival would be maximised where k_1 intersects k_3 at $\varphi \approx 35.3^\circ$ (or equivalently due to symmetry, where k_1 intersects k_2 at $\varphi \approx 325^\circ$).

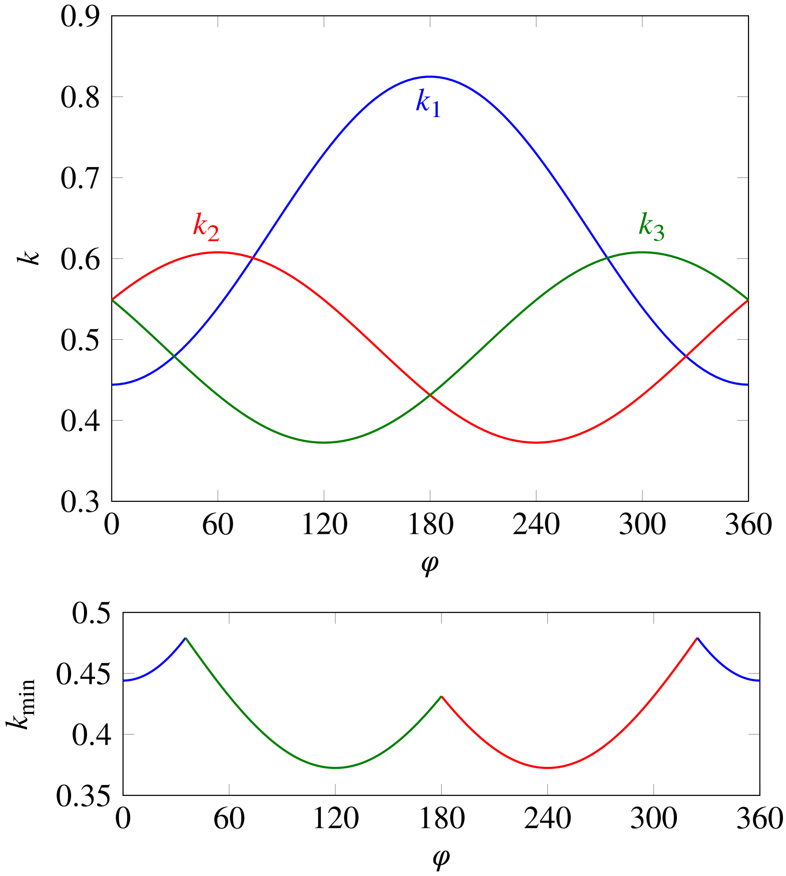

Figure 8: Graph of time of survival k (seconds) against φ (degrees), if v_1 were instead 20

4 Conclusion and Evaluation

In this paper, it has been ascertained that a single raptor travelling at v_r m s^{-1} pursuing a human travelling at v_h m s^{-1}, beginning D_0 m from and at an angle of θ_0, the human will be caught in $D_0\left(\frac{v_\text{r}-v_\text{h}\cos\theta_0}{{v_\text{r}}^2-{v_\text{h}}^2}\right)$ s, provided that $\frac{v_\text{r}}{2v_\text{h}}-\frac{1}{2}>0$. Based on this, it is concluded that, in the specific situation outlined in the introduction, considering all three raptors, the human would maximise their overall time of survival by running directly towards the top raptor.

While solving the question, a number of assumptions were made regarding the nature of the answers, namely the continuity and differentiability of $D(t)$ and $\cos\theta(t)$, justified with reference to the physical context of the questions. No other proof is offered for these assumptions, and if they are false, the results of this paper will be invalid. However, given the physical context, it is believed that these assumptions are reasonable.

Also, when $\theta_0 = 0°$ or $180°$, i.e. when the human runs directly towards or away from a raptor, the method outlined is invalid (as θ and hence $\cos\theta$ do not change with D). In these cases however, k is trivial to determine, and is simply $\frac{D_0}{v_r+v_k}$ or $\frac{D_0}{v_r-v_k}$ respectively, as correctly predicted by Equation 10, so the final results of this paper are unaffected.

Some other issues arise with the method taken in this paper of considering the human and raptors as infinitesimally small points, which can be seen, for example, in the result that $\cos\theta_k = -1$, i.e. that at the point of interception, the raptor always approaches the human from behind (except in the special case when $\theta_0 = 0$). This is not an intuitive result, as one would expect that in the real world, the raptor may well intercept the human from the front. This appears to be a result of requiring D to be precisely 0 before considering the human to have been intercepted, whereas in reality, the human merely needs to be within the raptor's reach to be devoured.

Similarly, this requirement for D to be 0 means that when $\frac{v_r}{2v_h} - \frac{1}{2} \leq 0$, i.e. when $v_h \geq v_r$, then k is undefined irrespective of θ_0 (except when $\theta_0 = 0$), as Equation 9 will cause zero to be raised to a negative power or to the power zero, hence suggesting that the human will never be caught by the raptor. In reality, there would be a small range of values of θ_0 where the human would nevertheless run sufficiently close to the raptor to be caught. An extension to this paper could be to consider these factors, or even investigate the range of angles θ_0 where the human would be caught even if $v_h \geq v_r$.

Another extension may be to consider other variations on the raptors' motion, for example, if v_r is a function of time, as would be the case if the raptors accelerate over time up to a given maximum speed, as in the first question in the comic. The general method outlined (solving the system of differential equations given by Equations 2 and 4) is applicable to such interpretations, however more complex motion may require a different approach to solving the systems of differential equations. For example, if v_r is dependent on t, Equation 6 would not be a separable differential equation in $\cos\theta$ and D, so another method would be required.

Finally, it is noted that the results of this paper, as shown in Figure 7, appear to be very similar, if not identical to a previous solution by Campbell [6], however Campbell uses the different interpretation of assuming that the raptors move in a straight line to intercept the human in the least time. This is a surprising result, and it may be an interesting extension to further investigate this phenomenon to ascertain whether this is a general result, and whether it can be applied to other, more complex, situations like that outlined above. If so, the technique may have significance for mathematical analysis of other pursuit curves with more realistic applications.

5 References

[1] MUNROE, Randall. *About* [online]. xkcd, n.d [viewed 2016-06-13]. Available from: `https://xkcd.com/about/`.

[2] MUNROE, Randall. *Substitute* [online graphic]. xkcd [viewed 2016-06-13]. Available from: `https://xkcd.com/135/` Licensed under the Creative Commons Attribution-NonCommercial 2.5 License.

[3] MUNROE, Randall. *Nerd Sniping* [online graphic]. xkcd, 2007 [viewed 2016-06-13]. Available from: `https://xkcd.com/356/`.

[4] LLOYD, Michael. Pursuit Curves. In: LLOYD, Michael (ed.). *Academic Forum 2006–07 / Number 24* [online]. United States: Henderson State University, 2006 [viewed 2016-06-13]. Available from: `http://hsu.edu/academicforum/2006-2007/2006-7AFPursuit.pdf`.

[5] MUNGAN, Carl E. A classic chase problem solved from a physics perspective. *European Journal of Physics*. 2005, **26**(6), 985–990. ISSN 1361-6404. Available from DOI: `10.1088/0143-0807/26/6/005`.

[6] CAMPBELL, Colin. *Raptor solution* [online]. United States: Pennsylvania State University, 2008 [viewed 2016-06-13]. Available from: http://www.phys.psu.edu/~campbell/raptor_solution.pdf Also available from Scribd: https://www.scribd.com/doc/36501691/Raptor-Solution.